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Exact conserved quantities on the cylinder II: off-critical case

Davide Fioravanti
Department of Mathematical Sciences
South Road, Durham DH1 3LE, England, UK
E-mail: Davide.Fioravanti@durham.ac.uk

Marco Rossi
Department of Mathematics, Heriot-Watt University
Edinburgh EH14 4AS, Scotland, UK
E-mail: M.Rossi@ma.hw.ac.uk

Abstract: With the aim of exploring a massive family of models, the nonlinear integral equation for a quantum system consisting of left and right KdV equations coupled on the cylinder is derived from an integrable lattice field theory. The eigenvalues of the energy and of the transfer matrix (and of all the other local integrals of motion) are expressed in terms of the corresponding solutions of the nonlinear integral equation. The family of models turns out to correspond to the $\Phi_{(1,3)}$ perturbation of Conformal Field Theories. The analytic and asymptotic behaviours of the transfer matrix are studied and given.

Keywords: Conformal and W Symmetry, Integrable Field Theories, Bethe Ansatz.

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Contents

1. Introduction

The computation of the eigenvalues of the conserved charges — in particular the local ones — is a very important issue in the study of $1+1$ dimensional Integrable Field Theories. For instance, the local integrals of motion have recently revealed to enter the form of off-critical null-vectors [1]. Among the various methods of computing, a possible one uses a discretisation of the space and the diagonalisation of the discrete transfer matrix by means of a Bethe Ansatz technique. The problem of recovering the continuous limit of such eigenvalues can be tackled by deriving non-linear integral equations, in the ductile form introduced by Destri and de Vega [2] for the ground state of sine-Gordon theory. Actually, similar types of non-linear integral equations were first discovered in the critical 6- and 19-vertex models by [3]. We will follow the scheme of [2] since it deals with a massive case very similar to ours and we may also simply gerealise our treatment to the excited states along the lines of several works (cfr. e.g. [4]).

Actually, our previous article [5] severely grounds the analysis of the present work, for it deals with recovering the (continuous) conformal field theory limit from the lattice (m)KdV theory (the quantum Lax operator has been written down by Kundu in [6]). It also provides us the natural ingredients to bring forward our analysis out of criticality, in an unified scenario. In this article we address the problem of performing the continuous limit of the eigenvalues of the massive transfer matrices in [7], by finding the nonlinear integral equation determining the counting function. In particular, exact expressions of all the local integrals of motion are given as nonlinear functionals of the counting function. Our approach bases on a non-ultralocal lattice field theory and therefore the continuum limit problem is different from that solved in [2, 4], since these papers used a lattice approach to the twisted sine-Gordon model based on an inhomogeneous spin $+1/2$ chain (with a suitably added twist). Moreover, this spin $+1/2$ chain discretisation does not furnish a transfer matrix for the sine-Gordon model (encoding all the conserved charges), whereas our construction is naturally endowed with an off-critical transfer matrix. On the other hand, our lattice model satisfies a braided version of the Yang-Baxter algebra [8, 6].

2. The nonlinear integral equation for the vacuum sector

The basic objects of the off-critical theory have been already introduced in the conformal case [5]. Therefore, we start by recalling the introduction of the mass [7], realised through coupling two KdV theories via dishomogeneities. More precisely, we considered on a lattice of length R, spacing Δ and number of sites $N = R/\Delta$, $N \in 4\mathbb{N}$, the following monodromy matrices

$$\mathbf{M}(\alpha) \equiv e^{-\frac{i\pi\beta^2}{2}} \prod_{i=1}^{\overleftarrow{N/4}} \bar{L}_{4i}(\alpha+\Theta)\bar{L}_{4i-1}(\alpha+\Theta)L_{4i-2}(\alpha-\Theta)L_{4i-3}(\alpha-\Theta)\,, \tag{2.1}$$

$$\mathbf{M}'(\alpha) \equiv e^{\frac{i\pi\beta^2}{2}} \prod_{i=1}^{\overleftarrow{N/4}} L_{4i}(\alpha-\Theta)L_{4i-1}(\alpha-\Theta)\bar{L}_{4i-2}(\alpha+\Theta)\bar{L}_{4i-3}(\alpha+\Theta)\,, \tag{2.2}$$

with coupling constant $0 \leq \beta^2 \leq 1$ and dishomogeneity Θ real. In these formulæ left Lax operators $L_m(\alpha)$ and right Lax operators $\bar{L}_m(\alpha)$ are given by

$$L_m(\alpha) \equiv \begin{pmatrix} e^{-iV_m^-} & e^{\alpha}e^{iV_m^+} \\ e^{\alpha}e^{-iV_m^+} & e^{iV_m^-} \end{pmatrix}, \qquad \bar{L}_m(\alpha) \equiv \begin{pmatrix} e^{-iV_m^+} & e^{-\alpha}e^{iV_m^-} \\ e^{-\alpha}e^{-iV_m^-} & e^{iV_m^+} \end{pmatrix}, \tag{2.3}$$

where $V_m^{\pm}$ are the discretised quantum counterparts of the mKdV variables v (left case) and $\bar{v}$ (right case) and satisfy the nonultralocal commutation relations

$$[V_m^{\pm}\,,\, V_n^{\pm}] = \pm\frac{i\pi\beta^2}{2}(\delta_{m-1,n} - \delta_{m,n-1})\,, \tag{2.4}$$

$$[V_m^{+}\,,\, V_n^{-}] = -\frac{i\pi\beta^2}{2}(\delta_{m-1,n} - 2\delta_{m,n} + \delta_{m,n-1})\,, \tag{2.5}$$

and the periodicity conditions $V_{m+N}^{\pm} = V_m^{\pm}$. Actually, the left Lax operator of 2.3 has been introduced by Kundu in [6] to define a (m)KdV monodromy matrix satisfying the braided Yang-Baxter algebra. In addition, we have proved in [7] that, as a consequence of (2.4, 2.5), monodromy matrices (2.1) and (2.2) satisfy braided Yang-Baxter equations. Moreover, the diagonalisation of the corresponding transfer matrices [7] is based on the Bethe equations

$$e^{-\varepsilon 2i\pi\beta^2 l} \prod_{\substack{r=1 \\ r\neq s}}^{l} \frac{\sinh(\alpha_s-\alpha_r+i\pi\beta^2)}{\sinh(\alpha_s-\alpha_r-i\pi\beta^2)} = \left[\frac{\sinh\left(\alpha_s+\Theta-\frac{i\pi\beta^2}{2}\right)\sinh\left(\alpha_s-\Theta-\frac{i\pi\beta^2}{2}\right)}{\sinh\left(\alpha_s+\Theta+\frac{i\pi\beta^2}{2}\right)\sinh\left(\alpha_s-\Theta+\frac{i\pi\beta^2}{2}\right)}\right]^{N/4}, \tag{2.6}$$

in which l is the number of Bethe roots and ε is equal to $+1$ for (2.1) and to -1 for (2.2). Correspondingly, the eigenvalues on Bethe states of the transfer matrices are

$$\mathbf{\Lambda}_N(\alpha) = \mathbf{\Lambda}_N^{+}(\alpha) + \mathbf{\Lambda}_N^{-}(\alpha)\,, \tag{2.7}$$

where

$$\mathbf{\Lambda}_N^{\pm}(\alpha) = e^{\mp\varepsilon i\pi\beta^2 l} \prod_{r=1}^{l} \frac{\sinh(\alpha-\alpha_r \pm i\pi\beta^2)}{\sinh(\alpha-\alpha_r)}\rho_N^{\pm}(\alpha) \tag{2.8}$$

and

$$\rho_N^{\pm}(\alpha) = e^{-\frac{\Theta N}{2}} \left[4 \sinh\left(\Theta - \alpha \mp \frac{i\pi\beta^2}{2}\right) \sinh\left(\Theta + \alpha \pm \frac{i\pi\beta^2}{2}\right)\right]^{N/4} . \tag{2.9}$$

Comparison with [9, equation (2.18)] shows that, after the identifications $2\cosh 2\Theta = 1/S$, $\alpha_s = \lambda_s - i\pi/2$ and the substitution $\beta^2 \to \beta^2/8\pi$, Bethe equations (2.6) coincide with the Bethe equations for the (space-time) lattice sine-Gordon model [10] if they were twisted by a dynamical factor $e^{-\varepsilon 2i\pi\beta^2 l}$.

Now we want to summarise all the Bethe equations (2.6) into a single nonlinear integral equation. As a preliminary, we define the function, analytic in the strip $|\mathrm{Im}\, x| < \min\{\zeta, \pi - \zeta\}$, $0 \le \zeta < \pi$,

$$\phi(x,\zeta) \equiv i \ln \frac{\sinh(i\zeta + x)}{\sinh(i\zeta - x)} , \tag{2.10}$$

which allows us to define the *counting function*

$$Z_N(x) \equiv \frac{N}{4}\left[\phi\left(x+\Theta, \frac{\pi\beta^2}{2}\right) + \phi\left(x - \Theta, \frac{\pi\beta^2}{2}\right)\right] + \sum_{r=1}^{l} \phi(x - \alpha_r, \pi\beta^2) + 2\pi\beta^2\varepsilon l . \tag{2.11}$$

In terms of the counting function the equations (2.6) assume the form

$$Z_N(\alpha_s) = \pi(2I_s - l - 1) , \qquad I_s \in \mathbb{N} . \tag{2.12}$$

Between the various solutions to the Bethe equations we single out the vacuum, i.e. the eigenstate with the minimum energy, for which α_s are real and $Z_N(\alpha_s)$ are equal to all the numbers of the form (2.12) between $Z_N(-\infty)$ and $Z_N(+\infty)$. From (2.11) it follows that these two extremes are

$$Z_N(\pm\infty) = \pm \frac{N}{4}(2\pi - 2\pi\beta^2) \pm l(\pi - 2\pi\beta^2) + 2\pi\beta^2 \varepsilon l . \tag{2.13}$$

The vacuum minimises the energy - which we will define in next section — when

$$\frac{1}{3} < \beta^2 < \frac{2}{3} , \tag{2.14}$$

and the "twist" $2\pi\beta^2\varepsilon l$ is small enough mod. 2π. Considering β^2 in the region (2.14), we now want to write an equation for the counting function associated to the vacuum in the continuous limit: $N \to +\infty$, R fixed. We firstly remark that (2.13) imply that in the continuous limit also $l \to +\infty$ and therefore, even if we define

$$\beta^2 = \frac{p}{p'} , \qquad p < p' \quad \text{coprimes} , \tag{2.15}$$

the last term in (2.11) is divergent. Therefore, we stick to the different subsequences

$$l = np' + \kappa , \qquad 0 \le \kappa \le p' - 1 , \tag{2.16}$$

where $n \to +\infty$ with κ fixed. Discarding in the definition of $Z_N(x)$ multiples of 2π we get:

$$Z_N(x) = \frac{N}{4}\left[\phi\left(x+\Theta, \frac{\pi}{2}\beta^2\right) + \phi\left(x-\Theta, \frac{\pi}{2}\beta^2\right)\right] + \sum_{r=1}^{l} \phi\left(x - \alpha_r, \pi\beta^2\right) + 2\pi\varepsilon\omega , \tag{2.17}$$

where ω is finite in the continuous limit (the double braces denote the fractional part):

$$\omega = \left\{\left\{\frac{p\kappa}{p'}\right\}\right\}. \tag{2.18}$$

Now, using standard techniques (cfr. e.g. [2]), one can rewrite a sum over the vacuum Bethe roots of a function f_N with no poles on a strip around the real axis as follows:[1]

$$\sum_{r=1}^{l} f_N(\alpha_r) = -\int_{-\infty}^{+\infty}\frac{dx}{2\pi}f_N'(x)Z_N(x) + 2\int_{-\infty}^{+\infty}\frac{dx}{2\pi}f_N'(x)\mathrm{Im}\ln\left[1+e^{iZ_N(x+i0)}\right]. \tag{2.19}$$

Applying (2.19) to the sum contained in (2.17) we get

$$\begin{aligned} Z_N(x) \;=\; & \frac{N}{4}\left[\phi\left(x+\Theta,\frac{\pi}{2}\beta^2\right)+\phi\left(x-\Theta,\frac{\pi}{2}\beta^2\right)\right]+2\pi\varepsilon\omega+ \\ & +\int_{-\infty}^{+\infty}\frac{dy}{2\pi}\phi'(x-y,\pi\beta^2)Z_N(y)-2\int_{-\infty}^{+\infty}\frac{dy}{2\pi}\phi'(x-y,\pi\beta^2)\mathrm{Im}\ln\left[1+e^{iZ_N(y+i0)}\right]. \end{aligned} \tag{2.20}$$

Let us define for sake of conciseness

$$\begin{aligned} K(x) &\equiv \frac{1}{2\pi}\phi'\left(x,\pi\beta^2\right), \qquad \Phi(x)\equiv\phi\left(x+\Theta,\frac{\pi}{2}\beta^2\right)+\phi\left(x-\Theta,\frac{\pi}{2}\beta^2\right), \\ L_N(x) &\equiv \mathrm{Im}\ln\left[1+e^{iZ_N(x+i0)}\right]. \end{aligned} \tag{2.21}$$

In these notations the Fourier transform of relation (2.20) is

$$\hat{Z}_N(k) = \frac{N}{4}\frac{\hat{\Phi}(k)}{1-\hat{K}(k)} - 2\frac{\hat{K}(k)}{1-\hat{K}(k)}\hat{L}_N(k) + 4\pi^2\varepsilon\omega\frac{\delta(k)}{1-\hat{K}(k)}, \tag{2.22}$$

where we have adopted the convention $\hat{f}(k) \equiv \int_{-\infty}^{+\infty} dx e^{-ikx} f(x)$ about the Fourier transform. Explicitly, the Fourier transforms of the first two functions in (2.21) are respectively

$$\hat{K}(k) = \frac{\sinh k\left(\frac{\pi}{2}-\pi\beta^2\right)}{\sinh\frac{\pi}{2}k}, \qquad \hat{\Phi}(k) = \frac{4\pi}{ik}\frac{\sinh k\left(\frac{\pi}{2}-\frac{\pi}{2}\beta^2\right)}{\sinh\frac{\pi}{2}k}\cos k\Theta. \tag{2.23}$$

Reparametrising the coupling constant β^2, introducing the kernel $G(\theta)$ and the renormalised twist $\tilde{\omega}$,

$$\gamma\equiv\pi(1-\beta^2), \qquad \hat{G}(k)\equiv-\frac{\hat{K}(k)}{1-\hat{K}(k)} = \frac{\sinh k\left(\frac{\pi}{2}-\gamma\right)}{2\cosh k\frac{\gamma}{2}\sinh k\left(\frac{\pi}{2}-\frac{\gamma}{2}\right)}, \qquad \tilde{\omega}\equiv\frac{\pi}{\beta^2}\omega, \tag{2.24}$$

equation (2.22) takes the compact form

$$\hat{Z}_N(k) = \frac{N}{4}\frac{\hat{\Phi}(k)}{1-\hat{K}(k)} + 2\hat{G}(k)\hat{L}_N(k) + 2\pi\varepsilon\tilde{\omega}\delta(k). \tag{2.25}$$

[1] We are fixing l to be even in the following (i.e. $e^{iZ_N(\alpha_s)} = -1$), as the case l odd can be treated along the same lines and in the end does not add new information.

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Now we want to Fourier anti-transform and then to perform the continuous limit: therefore, it is necessary to work out in such a limit the behaviour of the anti-transform of the first term in the r.h.s.

$$
\begin{aligned}
F_N(x) &\equiv \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx} \left(\frac{N}{4} \frac{\hat{\Phi}(k)}{1-\hat{K}(k)} \right) \\
&= \frac{\pi N}{4i} \int_{-\infty}^{+\infty} \frac{dk}{2\pi k} \left[e^{ik(x+\Theta)} + e^{ik(x-\Theta)} \right] \frac{\sinh k\left(\frac{\pi}{2} - \frac{\pi}{2}\beta^2\right)}{\sinh k \frac{\pi}{2}\beta^2 \cosh k \left(\frac{\pi}{2} - \frac{\pi}{2}\beta^2\right)} .
\end{aligned} \tag{2.26}
$$

This *driving term* may be evaluated through the residue theorem and it diverges in the continuous limit, unless Θ depends on N in a peculiar way. We sketch here the evaluation, since it differs from the previous ones [2]–[4] (even from the conformal case [5]). All the poles of the integrand lie on the imaginary axis and, when $1/3 < \beta^2 < 2/3$, the poles with smallest modulus are $k = 0, \pm i\frac{\pi}{\gamma}$. If in the continuous limit $\Theta \to +\infty$, we evaluate for fixed x the integral in (2.26) containing $e^{ik(x+\Theta)}$ by choosing a semi-circle contour in the upper k-complex half-plane with a small semicircle avoiding the pole at $k = 0$. Since the contributions from the other poles are damped by an exponential of Θ, the leading term is given by the residue at the pole with the smallest nonzero modulus, i.e. $k = i\pi/\gamma$, minus the contribution of the semicircle around $k = 0$, whose value is linear in N: $\frac{\pi N(1-\beta^2)}{8\beta^2}$. Analogously, in the evaluation of the integral containing $e^{ik(x-\Theta)}$ we close the contour — containing a semicircle avoiding $k = 0$ — in the lower k-complex half-plane. Now, the leading contribution is given by the pole at $k = -i\pi/\gamma$, minus the contribution of the semicircle around $k = 0$, whose value $\left(-\frac{\pi N(1-\beta^2)}{8\beta^2}\right)$ cancels the previous one. Adding the two residue contributions we obtain (up to terms $o(N^0) \to 0$ when $N \to +\infty$):

$$
N \to +\infty \, , \qquad F_N(x) = N e^{-\frac{\pi}{\gamma}\Theta} \frac{\sinh \frac{\pi}{\gamma} x}{\sin \frac{\pi\beta^2}{2(1-\beta^2)}} + o(N^0) = mR \sinh \frac{\pi}{\gamma} x + o(N^0) \, , \tag{2.27}
$$

where we have supposed a logarithmic divergence of Θ, via a constant mass parameter m (in such a way that mR is dimensionless):

$$
\Theta = -\frac{\gamma}{\pi} \ln \left[\frac{mR}{N} \sin \frac{\pi\beta^2}{2(1-\beta^2)} \right] . \tag{2.28}
$$

With this result in mind it is easy to write an equation satisfied by the counting function in the continuous limit, $Z(x)$, from relation (2.25):

$$
Z(x) = mR \sinh \frac{\pi}{\gamma} x + 2 \int_{-\infty}^{+\infty} dy \, G(x-y) \mathrm{Im} \ln \left[1 + e^{iZ(y+i0)} \right] + \varepsilon\tilde{\omega} \, . \tag{2.29}
$$

This *nonlinear integral equation* describes the $\tilde{\omega}$-vacuum of the theory at the different values of $\tilde{\omega}$.

Remark 1. In writing (2.22) we have paid the necessary attention to the zero modes of the operator to invert. Besides, we could more correctly derive equation (2.29) working out our calculations on the derivative of $Z(x)$ [4], but we think that our derivation is simpler, once all the subtleties are under control.

Remark 2. Equation (2.29) has been found for $1/3 < \beta^2 < 2/3$ and small $\tilde{\omega}$. However, it can be considered without problems in the whole region $0 < \beta^2 < 1$ and $\tilde{\omega} \geq 0$, defining by analytical continuation a state of the continuous theory which minimises the energy.

Remark 3. Equation (2.29) with $\tilde{\omega} = 0$ would also describe the vacuum of the continuous theory from the (space-time) lattice sine-Gordon model [10, 9], with lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 + \frac{m_0^2}{8\pi\beta^2}\cos\sqrt{8\pi}\beta\varphi\,, \tag{2.30}$$

where $m_0 = 2\sqrt{2}(R/N)^{-1}(\cosh 2\Theta)^{-1/2}$ is a bare mass.[2] Hence, there is a sort of interchange, given by the first of (2.24), between the coupling constant β in the lagrangian and γ in the equation (2.29).

3. Local charges and transfer matrix

The continuous limit of a sum like (2.19) can be expressed, once (2.29) has been put in, in terms of only the counting function $Z(x)$ (and the continuous limit of $f_N(x), f(x)$):

$$\lim_{\substack{N\to+\infty \\ N\Delta=R}} \sum_{r=1}^{l} f_N(\alpha_r) = f_b + \int_{-\infty}^{+\infty}\frac{dx}{\pi}J_f(x)\mathrm{Im}\ln\left[1+e^{iZ(x+i0)}\right]. \tag{3.1}$$

We have rearranged the r.h.s. in order to separate the bulk contribution

$$f_b = \lim_{N\to+\infty}\left[-\int_{-\infty}^{+\infty}\frac{dx}{2\pi}f'_N(x)\left(mR\sinh\frac{\pi}{\gamma}x + \varepsilon\tilde{\omega}\right)\right], \tag{3.2}$$

from the finite size corrections depending on the linear functional

$$J_f(x) = \lim_{N\to+\infty}\int_{-\infty}^{+\infty}\frac{dk}{2\pi}e^{ikx}ik\hat{f}_N(k)\hat{J}(k)\,, \quad \hat{J}(k) = \frac{\sinh k\frac{\pi}{2}}{2\cosh k\frac{\gamma}{2}\sinh k\left(\frac{\pi}{2}-\frac{\gamma}{2}\right)}. \tag{3.3}$$

Formula (3.1) allows to calculate exactly state depending quantities of a Quantum Field Theory on cylinder. In particular this formula applies effectively to the calculation of the energy of the $\tilde{\omega}$-vacuum. We build up the hamiltonian operator in tight analogy with the same construction in (space-time) lattice sine-Gordon model [10]

$$\begin{aligned} H \equiv -\frac{1}{4\Delta\sin\pi\beta^2\cosh 2\Theta}\Bigg[& e^{-2\alpha}\frac{\partial}{\partial\alpha}G_N^-(\alpha)\Big|_{\alpha=-\Theta-\frac{i\pi\beta^2}{2}} + e^{-2\alpha}\frac{\partial}{\partial\alpha}G_N^+(\alpha)\Big|_{\alpha=-\Theta+\frac{i\pi\beta^2}{2}} - \\ & - e^{2\alpha}\frac{\partial}{\partial\alpha}G_N^-(\alpha)\Big|_{\alpha=\Theta-\frac{i\pi\beta^2}{2}} - e^{2\alpha}\frac{\partial}{\partial\alpha}G_N^+(\alpha)\Big|_{\alpha=\Theta+\frac{i\pi\beta^2}{2}}\Bigg], \end{aligned} \tag{3.4}$$

[2] From (2.28) we have the renormalised mass parameter $m = \left[\frac{m_0}{4}\left(\frac{R}{N}\right)^{\beta^2}\right]^{\frac{\pi}{\gamma}} \Big/ \sin\frac{\pi\beta^2}{2(1-\beta^2)}$ in the continuous limit.

where we have defined $G_N^{\pm}(\alpha) \equiv \ln\left[\rho_N^{\pm}(\alpha)^{-1}\mathbf{\Lambda}_N(\alpha)\right]$. Using (2.7)–(2.9, (3.4), the eigenvalues of H for a general Bethe state read as a sum on Bethe roots, $\sum_{r=1}^{l} h(\alpha_r, N, R)$, where

$$
\begin{aligned}
h(x,N,R) &= -\frac{1}{4\Delta \sin \pi\beta^2 \cosh 2\Theta} \times \\
&\quad \times \left[-\frac{e^{\Theta - x - \frac{i\pi\beta^2}{2}}}{\sinh\left(x+\Theta+\frac{3i\pi\beta^2}{2}\right)} + \frac{e^{\Theta - x + \frac{i\pi\beta^2}{2}}}{\sinh\left(x+\Theta+\frac{i\pi\beta^2}{2}\right)} + \right. \\
&\quad + \frac{e^{3\Theta - x - \frac{5i\pi\beta^2}{2}}}{\sinh\left(x-\Theta+\frac{3i\pi\beta^2}{2}\right)} - \frac{e^{3\Theta - x - \frac{3i\pi\beta^2}{2}}}{\sinh\left(x-\Theta+\frac{i\pi\beta^2}{2}\right)} - \frac{e^{\Theta - x + \frac{i\pi\beta^2}{2}}}{\sinh\left(x+\Theta-\frac{3i\pi\beta^2}{2}\right)} + \\
&\quad \left. + \frac{e^{\Theta - x - \frac{i\pi\beta^2}{2}}}{\sinh\left(x+\Theta-\frac{i\pi\beta^2}{2}\right)} + \frac{e^{3\Theta - x + \frac{5i\pi\beta^2}{2}}}{\sinh\left(x-\Theta-\frac{3i\pi\beta^2}{2}\right)} - \frac{e^{3\Theta - x + \frac{3i\pi\beta^2}{2}}}{\sinh\left(x-\Theta-\frac{i\pi\beta^2}{2}\right)} \right].
\end{aligned}
\tag{3.5}
$$

When $1/3 < \beta^2 < 2/3$, extending the results of [11] to $\tilde{\omega} \neq 0$, we are able to show that the $\tilde{\omega}$-vacuum is indeed the eigenstate with the lowest eigenvalue. Moreover, when the twist $\tilde{\omega} = 0$ this is the energy of the (space-time) lattice sine-Gordon model [10].

After Fourier transforming [12]

$$
\begin{aligned}
ik\hat{h}(k,N,R) &= \frac{ie^{2\Theta}}{\Delta \cosh 2\Theta \sin \pi\beta^2} \frac{\pi k \sinh k\frac{\pi}{2}\beta^2}{\sinh k\frac{\pi}{2}} \times \\
&\quad \times \left[\sin\left(k\Theta - \pi\beta^2\right) e^{-k\frac{\pi}{2}+k\pi\beta^2} - \sin\left(k\Theta + \pi\beta^2\right) e^{k\frac{\pi}{2}-k\pi\beta^2}\right],
\end{aligned}
\tag{3.6}
$$

we are left with the evaluation of $J_h(x)$ (3.3) in the continuous limit

$$
\begin{aligned}
J_h(x) &= \lim_{N\to\infty} \left\{ \frac{Nie^{2\Theta}}{2R\cosh 2\Theta \sin \pi\beta^2} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx} \frac{\pi k}{\cosh k\frac{\gamma}{2}} \times \right. \\
&\quad \left. \times \left[\sin\left(k\Theta - \pi\beta^2\right) e^{-k\frac{\pi}{2}+k\pi\beta^2} - \sin\left(k\Theta + \pi\beta^2\right) e^{k\frac{\pi}{2}-k\pi\beta^2}\right] \right\}.
\end{aligned}
\tag{3.7}
$$

Again, because of relation (2.28) we can calculate exactly this limit by using the residue method (closing the contour of integration in the upper or in the lower k-complex half-plane). Collecting the finite contributions we obtain a simple behaviour

$$
J_h(x) = -4m \frac{\sin\left[\frac{\pi}{2}\frac{1-2\beta^4}{1-\beta^2}\right] \sin\frac{\pi\beta^2}{2(1-\beta^2)}}{(1-\beta^2)^2 \sin \pi\beta^2} \sinh\frac{\pi}{\gamma}x\,. \tag{3.8}
$$

Hence, the finite size correction $E - E_b$ to the energy of the $\tilde{\omega}$-vacuum is

$$
E - E_b = -4m \frac{\sin\left[\frac{\pi}{2}\frac{1-2\beta^4}{1-\beta^2}\right] \sin\frac{\pi\beta^2}{2(1-\beta^2)}}{(1-\beta^2)^2 \sin \pi\beta^2} \int_{-\infty}^{+\infty} \frac{dx}{\pi} \sinh\frac{\pi}{\gamma}x \,\mathrm{Im}\ln\left[1 + e^{iZ(x+i0)}\right]. \tag{3.9}
$$

As far as the $\tilde{\omega}$-vacuum state is concerned, this formula has been derived within a twisted lattice sine-Gordon model and hence proves the quantum equivalence with the approaches

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to the sine-Gordon model based on spin $+1/2$ XXZ chain [2, 4]. However, a different renormalisation of the mass occurs in (3.9).

We now want to compute the continuous limit of the eigenvalues of the transfer matrix (2.7) proposed in [7] as description of conformal field theories perturbed by $\Phi_{(1,3)}$. To make formula (3.1) useful, we shall define $\mathbf{F}_N^{\pm}(\alpha) \equiv \ln \mathbf{\Lambda}_N^{\pm}(\alpha)$. Let us firstly concentrate on $\mathbf{F}_N^{+}(\alpha)$:

$$\begin{aligned} \mathbf{F}_N^{+}(\alpha) &= \sum_{r=1}^{l} \left[\ln \frac{\sinh(\alpha - \alpha_r + i\pi\beta^2)}{\sinh(\alpha - \alpha_r)} - i\pi\varepsilon\beta^2 \right] + \\ &+ \frac{N}{4} \left[\ln 2 \sinh \left(\Theta - \alpha - \frac{i\pi\beta^2}{2} \right) + \ln 2 \sinh \left(\Theta + \alpha + \frac{i\pi\beta^2}{2} \right) - 2\Theta \right]. \end{aligned} \tag{3.10}$$

Now we restrict ourselves to the $\tilde{\omega}$-vacua, although generalisations to excited states can be worked out along the developments of [4]. The last addendum in (3.10) gives a contribution whose continuous limit is zero if $0 < \beta^2 < 1/2$ and infinity if $1/2 < \beta^2 < 1$. Therefore, we shall regularise $\mathbf{F}^+(\alpha)$ for $1/2 < \beta^2 < 1$ defining it as the analytic continuation of $\mathbf{F}^+(\alpha)$ with $0 < \beta^2 < 1/2$. With this regularisation in mind we obtain from (3.1)

$$\mathbf{F}^+(\alpha) = \mathbf{F}_b^+(\alpha) + \int_{-\infty}^{+\infty} \frac{dx}{\pi} \left[\int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx} ik \hat{f}^+(k,\alpha) \hat{J}(k) \right] \mathrm{Im} \ln \left[1 + e^{iZ(x+i0)} \right] - \varepsilon i\pi\omega \,, \tag{3.11}$$

where $\mathbf{F}_b^+(\alpha)$ is the bulk contribution, $\hat{J}(k)$ is given by (3.3) and

$$f^+(x,\alpha) = \ln \frac{\sinh\left(\alpha - x + i\pi\beta^2\right)}{\sinh(\alpha - x)} \,. \tag{3.12}$$

The twist ω comes out as in [5]. Since $\mathbf{F}^+(\alpha)$ (3.10) has the periodicity property $\mathbf{F}^+(\alpha + i\pi) = \mathbf{F}^+(\alpha)$, it is sufficient to study $\mathbf{F}^+(\alpha)$ in the strip $\gamma - \pi < \mathrm{Im}\alpha < \gamma$. Comparing (3.11) with relation (5.6) of [5], we remark that the second term in the r.h.s. is formally the same. Therefore, we can use the results of that paper: from formulæ (5.16, 5.24) of [5] we obtain that, if $0 < \mathrm{Im}\alpha < \gamma$,

$$\mathbf{F}^+(\alpha) = \mathbf{F}_b^+(\alpha) - \int_{-\infty}^{+\infty} \frac{dx}{\gamma} \frac{1}{\sinh \frac{\pi}{\gamma}(x-\alpha)} \mathrm{Im} \ln \left[1 + e^{iZ(x+i0)} \right] - \varepsilon i\pi\omega \,, \tag{3.13}$$

and that, if $\gamma - \pi < \mathrm{Im}\alpha < 0$,

$$\begin{aligned} \mathbf{F}^+(\alpha) &= \mathbf{F}_b^+(\alpha) - \varepsilon i\pi\omega + \int_{-\infty}^{+\infty} \frac{dx}{\pi} \left\{ \int_{-\infty}^{+\infty} dk e^{ik\left(x - \alpha - \frac{i\pi\beta^2}{2}\right)} \frac{i \sinh k\frac{\gamma}{2}}{2 \cosh k\frac{\gamma}{2} \sinh k\left(\frac{\pi}{2} - \frac{\gamma}{2}\right)} \right\} \times \\ &\times \mathrm{Im} \ln \left[1 + e^{iZ(x+i0)} \right]. \end{aligned} \tag{3.14}$$

On the other hand, from definitions (3.2) and (3.12) it follows that

$$\mathbf{F}_b^+(\alpha) = \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \frac{\sinh i\pi\beta^2}{\cosh\left(\alpha - x - \frac{i\pi}{2}\right) \cosh\left(\alpha - x - \frac{i\pi}{2} + i\pi\beta^2\right)} \left[mR \sinh \frac{\pi}{\gamma} x + \varepsilon\tilde{\omega} \right]. \tag{3.15}$$

This integral is finite only for $0 < \beta^2 < 1/2$. If $0 < \mathrm{Im}\alpha < \gamma$ its value is

$$\mathbf{F}_b^+(\alpha) = -mR\,\mathrm{ctg}\,\frac{\pi^2}{2\gamma}\cosh\frac{\pi}{\gamma}\alpha + i\varepsilon\pi\omega\,, \tag{3.16}$$

whereas, if $\gamma - \pi < \mathrm{Im}\alpha < 0$ it equals

$$\mathbf{F}_b^+(\alpha) = -mR\,\frac{1}{\sin\frac{\pi^2}{2\gamma}}\cosh\left[\frac{\pi}{\gamma}\left(\alpha + i\frac{\pi}{2}\right)\right] + i\varepsilon\pi\omega - i\varepsilon\tilde{\omega}\,. \tag{3.17}$$

Let us now consider $\mathbf{F}^-(\alpha)$ which reads from (2.7)–(2.9)

$$\begin{aligned}\mathbf{F}_N^-(\alpha) &= \sum_{r=1}^{l}\left[\ln\frac{\sinh(\alpha-\alpha_r-i\pi\beta^2)}{\sinh(\alpha-\alpha_r)} + i\pi\varepsilon\beta^2\right] + \\ &+ \frac{N}{4}\left[\ln 2\sinh\left(\Theta - \alpha + \frac{i\pi\beta^2}{2}\right) + \ln 2\sinh\left(\Theta + \alpha - \frac{i\pi\beta^2}{2}\right) - 2\Theta\right]. \end{aligned}\tag{3.18}$$

Again, in the continuous limit the last term is zero in the interval $0 < \beta^2 < 1/2$ and we use our regularisation procedure. According to the latter we obtain a relation

$$\mathbf{F}^-(\alpha) = -\mathbf{F}^+\left(\alpha - i\pi\beta^2\right), \tag{3.19}$$

which gives $\mathbf{F}^-(\alpha)$ for any complex α. From the knowledge of $\mathbf{F}^\pm(\alpha)$ it is simple to reconstruct $\mathbf{\Lambda}(\alpha) = \exp[\mathbf{F}^+(\alpha)] + \exp[\mathbf{F}^-(\alpha)]$ in all the α-complex plane.

From the classical results [13] we can argue that the coefficients of the asymptotic expansion of $\ln\mathbf{\Lambda}(\alpha)$ are proportional to the local integrals of motion in involution $\mathbf{I}_{2n+1}^{\mp}$ of perturbed conformal field theories. Therefore, using the same techniques as in [5], we expand $\ln\mathbf{\Lambda}(\alpha)$ for $\mathrm{Re}\alpha \to \pm\infty$ in the strip $\max\{-\pi/2, -\gamma\} < \mathrm{Im}\alpha < \min\{\pi/2, \gamma\}$ (which is only a technical assumption):

$$\begin{aligned}\ln\mathbf{\Lambda}(\alpha) &\doteq -mR\,\mathrm{ctg}\,\frac{\pi^2}{2\gamma}\cosh\frac{\pi}{\gamma}\alpha \pm \\ &\pm\frac{2}{\gamma}\sum_{n=0}^{+\infty} e^{\mp\frac{\pi}{\gamma}\alpha(2n+1)}\int_{-\infty}^{+\infty} dx\, e^{\pm\frac{\pi}{\gamma}x(2n+1)}\,\mathrm{Im}\ln\left[1 + e^{iZ(x+i0)}\right]. \end{aligned}\tag{3.20}$$

Now, we need to compare this expression with the conformal analogous (5.48) of [5] and to consider the conformal normalisation coefficients c_n ([5, (5.50)]), to write down

$$\ln\mathbf{\Lambda}(\alpha) \doteq -mR\,\mathrm{ctg}\,\frac{\pi^2}{2\gamma}\cosh\frac{\pi}{\gamma}\alpha \pm \frac{2}{\pi}\sum_{n=0}^{+\infty} e^{\mp\frac{\pi}{\gamma}\alpha(2n+1)} c_{n+1}\mathbf{I}_{2n+1}^{\mp}\,, \tag{3.21}$$

where the local integrals of motion have the exact expressions ($\chi = \mp 1$):

$$\mathbf{I}_{2n+1}^{\chi} = \frac{1}{c_{n+1}}\frac{\pi}{\gamma}\int_{-\infty}^{+\infty} dx\, e^{-\chi\frac{\pi}{\gamma}x(2n+1)}\,\mathrm{Im}\ln\left[1 + e^{iZ(x+i0)}\right]. \tag{3.22}$$

Indeed, we have normalised them with respect to the conformal counterparts I_{2n+1} (cfr. [5, (5.51)]) so that in the conformal limit

$$\lim_{r\to 0}\left(\frac{r}{2A}\right)^{2n+1}\mathbf{I}_{2n+1}^{\chi=\varepsilon} = I_{2n+1}\,, \tag{3.23}$$

where $r = mR$ is a dimensionless parameter. Moreover, we have been able to build up a lattice field theory for which conjecture (3.31) of Bazhanov, Lukyanov and Zamolodchikov [14] — suggested by analogy with the conformal case [15] — holds. Nevertheless, the naïve scaling limit of our monodromy matrix [7] is absolutely different from the proposed (but not actually used) prescription given in [14]. On the other hand, we are in agreement with the finding of [14] derived using a brilliant development of the Thermodynamic Bethe Ansatz (TBA) technique in the case of scaling Lee-Yang model (cfr. e.g. [16], the first of [17] and references therein).

For $\beta^2 = 1/2$ the local integrals of motion (3.22) are explicitly given by

$$\mathbf{I}^{\chi}_{2n+1} = -2^{1-n}\pi^{2n}(n+1)(2n+1)\mathrm{Im}\int_{-\infty}^{+\infty} dx e^{-\chi 2(x+i\eta)(2n+1)} \ln\left\{1+e^{i[mR\sinh 2(x+i\eta)+2\pi\varepsilon\omega]}\right\}, \tag{3.24}$$

where $0 < \eta < \pi/2$. The choice $\eta = \pi/4$ gives

$$\mathbf{I}^{\chi}_{2n+1} = \chi(-1)^n 2^{1-n}\pi^{2n}(n+1)(2n+1)r^{-2n-1}\,\mathrm{Re}P_{2n+1}(a,r)\,, \tag{3.25}$$

where $a = -2\pi\varepsilon(\omega + 1/2)$ and we have introduced the rescaled charges

$$P_{2n+1}(a,r) = r^{2n+1}\int_0^{+\infty} d\theta\, \cosh\theta(2n+1)\, \ln\left[1 - e^{-r\cosh\theta - ia}\right], \tag{3.26}$$

which have a finite conformal limit ($r \to 0$). Their derivative with respect to r is written, after an integration by parts, in terms of modified Bessel functions,

$$\frac{dP_{2n+1}}{dr}(a,r) = r^{2n+1}\sum_{k=1}^{+\infty} e^{-iak}K_{2n}(kr)\,. \tag{3.27}$$

After integrating this formula and considering that the integration constant is fixed by the known [5] conformal limit (3.23), we would obtain an expression of $\mathbf{I}^{\chi}_{2n+1}$ as series of modified Bessel functions. Nevertheless, we prefere to differentiate (3.27) and write down a relation involving K_0 only [12]:

$$\left(\frac{1}{r}\frac{d}{dr}\right)^{2n+1}\mathrm{Re}P_{2n+1}(a,r) = \sum_{k=1}^{+\infty}\cos ak\, k^{2n}K_0(kr) = (-1)^n\frac{d^{2n}}{da^{2n}}\sum_{k=1}^{+\infty}\cos akK_0(kr)\,. \tag{3.28}$$

This expression allows us to simplify the form of the off-critical charges via a Schlömilch formula [12] ($\mathbf{C}$ is the Euler-Mascheroni constant):

$$\begin{aligned}\sum_{k=1}^{\infty} K_0(kr)\cos ka &= \frac{1}{2}\left(\mathbf{C} + \ln\frac{r}{4\pi}\right) + \frac{\pi}{2\sqrt{r^2+a^2}} + \\ &\quad + \frac{\pi}{2}\sum_{l=1}^{\infty}\left[\frac{1}{\sqrt{r^2+(2l\pi-a)^2}} + \frac{1}{\sqrt{r^2+(2l\pi+a)^2}} - \frac{1}{\pi l}\right].\end{aligned} \tag{3.29}$$

Term by term derivatives can be performed using the formula

$$\frac{d^{2n}}{da^{2n}}\frac{1}{\sqrt{x^2+a^2}} = \frac{\sum_{\substack{h=0\\h\in 2\mathbb{N}}}^{2n} b_h^{(2n)}a^h x^{2n-h}}{(x^2+a^2)^{2n+1/2}}\,, \qquad b_h^{(2n)} = \frac{(-1)^{n+h/2}\left[(2n-1)!!\right]^2}{h!}\prod_{\substack{k=0\\k\in 2\mathbb{N}}}^{h-2}(2n-k)^2\,. \tag{3.30}$$

We give the explicit result for the case $n = 0$,

$$\mathrm{Re}P_1(a,r) = \frac{r^2}{8}[2\mathbf{C} - 2\ln 4\pi + \ln r^2 - 1] + \frac{\pi}{2}\sqrt{r^2+a^2} - \frac{\pi}{2}|a| + \\ + \frac{\pi}{2}\sum_{l=1}^{\infty}\left[\sqrt{r^2+(2l\pi-a)^2} + \sqrt{r^2+(2l\pi+a)^2} - \frac{r^2}{2\pi l} - 4\pi l\right] - \pi^2 B_2\left(\frac{a}{2\pi}\right), \tag{3.31}$$

because in the cases $\omega = 0, \pm 1/2$ it reproduces known TBA results [18].

So far, we have studied the monodromies of $\mathbf{I}^{\chi}_{2n+1}$ on $r = mR$. Actually, it is also important to see how the charges depend on the twist ω. Let us extend (3.24) to semi-integers $n \geq 0$. Introducing the index $k \in \mathbb{N}$, $k \geq 1$, we find that the following differential equation holds:

$$\left(\frac{r\chi}{2\pi\varepsilon}\right)\frac{1}{2^{1/2}\pi(k+3)}\frac{d\mathbf{I}^{\chi}_{k+2}}{d\omega} + \left(\frac{r\chi}{2\pi\varepsilon}\right)\frac{\pi(k+2)}{2^{3/2}k(k+1)}\frac{d\mathbf{I}^{\chi}_{k}}{d\omega} - \mathbf{I}^{\chi}_{k+1} = 0\,. \tag{3.32}$$

In the conformal limit $r \to 0$, taking into account (3.23), we get the solution $I_{k+1} = -\varepsilon 2^{-1-k/2}B_{k+2}(\omega + 1/2)$.

4. Summary and outlook

Starting from the lattice theory of two coupled (m)KdV equations, we have found in the whole interval $0 < \beta^2 < 1$ the nonlinear integral equation which describes the vacuum sector and we are in the position to work out all features of excited states after [4]. The existence in our approach of a transfer matrix (which generates the local integrals of motion in involution) has allowed us to give expressions for its eigenvalues on the twisted vacua in terms of the solutions of the nonlinear integral equation and makes possible, in the next future, to find a deep link with the Thermodynamic Bethe Ansatz technique of Bazhanov, Lukyanov and Zamolodchikov [14]. We have found strong evidence that local charge eigenvalues coincide with the primary state eigenvalues of the local integrals of motion of $\Phi_{(1,3)}$ perturbed conformal field theories, the primary weight being linked to the twist. The expressions become explicit for $\beta^2 = 1/2$. Eventually, we want to emphasise that the wide generality of our construction [7, 5] makes possible its application, e.g., in the domain of two-dimensional supersymmetric field theories, like super KdV (cfr. for instance [19] and references therein). Also, a formulation of a off-critical TQ relation à la Baxter [20] with exact formulæ for Q is easily derivable.

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